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No. 2342: November 8, 1923

The Texas Mathematics Teachers' Bulletin

Volume IX, No. 1

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar

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Volume IX, No. 1

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This Bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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MATHEMATICS FACULTY OF THE UNIVERSITY OF TEXAS

ON MENTAL ARITHMETIC AND RELATED TOPICS

The writer recalls the lamentations of many friends and acquaintances over the fact that mental arithmetic is now practically a lost art. Men of practical affairs are particularly interested in the fact that our young men and women in these days and times do not have the nimbleness with figures that their forbears had. The college instructor of freshmen is very early struck with the realization that most of his students are not very adept in handling straight arithmetic. How often does the freshman use long division to divide a number by 7? The simplest squares are beyond their ken, the square roots of numbers without a pencil are *terra incognita.*

It may be asked why worry about mental arithmetic in the day of labor saving machinery such as computing machines, tables and slide rules. Why stimulate the development or rebirth of mental arithmetic by incorporating in the contests of the Interscholastic League a mental arithmetic test similar to the spelling and music memory contests. The first would be much less a test of memory than the latter two. Mr. Russell of the Central High School of Fort Worth is the author of this suggestion.

There are many benefits to be urged for a thorough grounding in numbers. It has become very fashionable of late to speak in large numbers (a modest sum of money expressed in Polish or German marks), but very few students have any adequate notions of magnitude. The relative magnitudes in centimeters of the diameters of an electron and the universe, the wave length of sodium and that of a very long wireless wave represent adventures into the realms of the very small and the very large. How big is 2 raised to the thirty second power? How big is 2 raised to the minus tenth power? Not only is the judgment as to the ratio of large and small lacking in students but it is necessary to point out as early as possible that "large" and "small" are relative terms and involve implicitly a ratio.

A number is large or small compared with some other number. Judgment as to size is one of the ends sought in number training. Should we expect that our students be able to state (without a moment's hesitation), the magnitude roughly of the product of 23.45 and 72.468, that this answer is surely greater than 14,000 and less than 24,000, and hence that the result of multiplication should yield five digits to the left of the decimal point. For surely the average pupil can be trained to see that the answer is between the product of 200 and 70 and that of 300 and 80. Similarly we may develop judgment as to the magnitude of the quotient of two numbers.

Few students are willing to admit that the combination

$$\frac{-2+\sqrt{11}}{6}$$

represents a number; that this strange group of symbols may be the result of solving a quadratic equation they may finally realize (though there must have been some mistake since the answer did not come out "even"). How big is this answer, if it were to be laid off in feet with a foot rule, how large a distance would it measure? Is it beyond the understanding of a high school student to realize that $\sqrt{11}$ is a number which cannot be expressed in terms of a whole number and a fraction such that the decimal part of the expression shall terminate at a definite place? That the "square root" process yields a number which for us represents an approximation to the value of $\sqrt{11}$ which differs from $\sqrt{11}$ by less than unity in the last decimal place which has been found.

The product of any number by a number like 25 or 125 or $16\frac{2}{3}$ *i.e.* of the type $\frac{10^n}{p}$ where n is any integer and p is one of the digits 1, 2, 3, . . . 9, may always be accomplished by moving the decimal point n places (to the right if n is positive and to the left if n is negative) and dividing by p. If the student can divide by any of the natural numbers up to 9, without using pencil and paper, he can do orally by means of this suggestion a multiplication which might otherwise be quite laborious and lengthy. A similar remark might be made concerning division by any number of the form $\frac{10^n}{p}$. In this case, multiply the given number by p and move the decimal point $-n$ places, *i.e.*, to the left if n is positive and to the right if n is negative.

The square of a binomial and the product of the sum and difference of two numbers afford an excellent opportunity for developing mental arithmetic. They furthermore permit of an unexcelled series of examples to stimulate interest in the early part of high school algebra, a stage of mathematics which is otherwise void of "practical applications." For example to find $(209)^2$, we perform mentally $(200+9)^2=(200)^2+(9)^2+2(200)\ (9)=43681$. If necessary in the early stages of practice these steps could be written out as above, but after a little they could be carried out orally. Similarly

$$(199)^2=(200-1)^2=(200)^2+(1)^2-2(200)\ (1)$$
$$=36{,}001.$$

and

$$(151)\ (149)=(150+1)\ (150-1)=(150)^2-(1)^2$$
$$=22{,}499.$$

The square root of a number can be obtained approximately by means of the equation

$$\sqrt{(1+x)}=1+(1/2)x+E$$

where E represents the error and is less than x^2 in numerical value, if x is less than one numerically. The following examples illustrate these methods:

$$\sqrt{1.02}=\sqrt{(1+.02)}=1+(1/2)\ (.02)+=1.01+$$

and the error is surely less than .0004,
and

$$\sqrt{0.98}=\sqrt{(1-.02)}=1+(1/2)\ (-.2)=0.99+$$

and again the error is surely less than .0004.

The reciprocal of a number can similarly be obtained by means of the equation

$$\frac{1}{1+x}=x+E$$

where E is numerically less than x^2 provided x is less than one numerically. The following example illustrates this method

$$1/1.005=1-.005=.995+$$

and the error is numerically less than .000025.

In conclusion, it may be pointed out that the ability to multiply two four digit numbers does not absorb all the possibilities of mental arithmetic, in fact, the writer has avowedly excluded from this discussion any performances with numbers which might savor of mental gymnastics of the acrobatic type.

H. J. ETTLINGER.

THE STRUCTURE OF THE UNIVERSE

The title which I have chosen for this paper is a rather pretentious one, perhaps more so than the present state of our knowledge warrants. Very rapid progress has been made in stellar astronomy during the last twenty years, however, and much information has been gathered bearing on the shape and size of the stellar universe, although the subject still bristles with conflicting views and unsolved problems.

The problem of the structure of the universe is largely that of the distances of the stars. The position of a star in the sky can be measured with great accuracy, and this gives its direction; but up till less than a hundred years ago we did not know the distance of a single one. The first successful measurement of the distance of a star was made by Bessel in 1838, by measuring the slight difference in its direction when the earth is on opposite sides of the sun; one-half this angular difference, *i.e.*, the angle subtended at the star by the radius of the earth's orbit, is called its parallax. Since the radius of the earth's orbit is known, the determination of the distance of the star from its parallax is a simple problem in trigonometry. The parallaxes of even the nearest stars, however, are so small and difficult to measure that up to the beginning of the present century only about eighty had been determined. Since then improved photographic methods have increased the number to more than a thousand, but the best we can hope for from this direct method is to get the distances of perhaps two thousand stars, out of the hundreds of millions that are visible in our telescopes.

A larger number of stars can be reached by using as our base line not the diameter of the earth's orbit, but the path traced out by the solar system as it moves through space. It is well established that the sun is moving toward a point in the constellation Hercules with a speed of about twelve miles a second, so if we measure the position of a star at a certain time and again after an interval of several years, we shall find it changed as a result of this motion.

This method cannot be used for an individual star, which is itself in motion, but if we take a large group of stars, for which we may assume that the individual motions cancel each other, then we can find the average distance of the group in this way.

The motion of a star across the sky, due to the combined effect of its own motion in space and the motion of the sun, is called its "proper motion"; it is usually measured in seconds of arc per year or per century. We are also able to determine by means of the spectroscope whether it is moving toward us or away from us, and at what speed. If a star is approaching us, the lines in its spectrum are shifted toward the violet end, and if it is receding from us they are shifted toward the red end, the amount of shift being proportional to the velocity of approach or recession. This velocity is called the "radial velocity," or "velocity in the line of sight," and its determination is independent of the distance of the star. If we know the distance, then we can translate the proper motion into linear speed (*e.g.*, into miles per second), and by combining this with the radial velocity we obtain both the magnitude and direction of the motion of the star through space.

The rapid progress of the last few years has been due to the discovery of certain indirect methods of determining distances. In 1917 W. S. Adams of the Mount Wilson Observatory found a relation between the absolute luminosity of a star and the relative intensity of certain lines in its spectrum. Hence in the case of any star of certain spectral types bright enough so that we can measure the intensities of the lines in its spectrum we can compute the absolute brightness, and then by comparing that with the apparent brightness we can get the distance, knowing that the luminosity varies inversely as the square of the distance. Results obtained in this way agree admirably with the distances measured by trigonometric methods in cases where both methods can be used.

Another valuable method makes use of what are called the Cepheid variable stars. These are stars of great absolute brightness whose light varies rapidly; the period is sometimes only a few hours, though it may be as much as several weeks. There is reason to suppose that they are vast masses of gas of low density, the variation being due to regular contraction and expansion of the gas. There are a number of these variables in the Lesser Magellanic Cloud, a star cluster in the southern part of the sky (too far south to be visible at this latitude), which looks like an isolated patch of the Milky Way. Miss Leavitt of the Harvard Observatory discovered that there is a definite relationship between the period and the apparent brightness of the variables in this Cloud. Since all the stars in the Cloud are at approximately the same distance from the sun, this implies a relation between the period of variation and the absolute brightness. A direct determination of the distances of the nearer Cepheid variables by trigonometric methods then made it possible to draw a curve giving the exact relation between the period of variation and the absolute brightness. Hence we are able to determine the distance of any Cepheid variable by observing its period, getting the corresponding absolute brightness from the curve, and then comparing that with its apparent brightness.

Another important type of variable stars is the eclipsing binary; here we have a double star whose components revolve about each other in orbits such that one star periodically passes between the other star and the earth, thus eclipsing it. Just from a study of the light variation of such a system it is possible to compute with considerable accuracy the absolute brightness of the two components and hence, knowing their apparent brightness, to obtain their distance.

These new methods enable us to extend our investigations much farther into space than was possible ten years ago, and have given us an entirely new conception of the size of the stellar universe. Previously our knowledge was confined almost exclusively to the stars in the neighborhood of the sun.

The nearest star is Alpha Centauri, about 25 trillion miles away. Stellar distances are so large that we usually state them in terms of "light-years"; light travels at the rate of 186,000 miles per second, and a light-year is the distance covered in one year by a ray of light moving at this speed; it is about six trillion miles. Another unit frequently used is the "parsec," which is the distance at which a star would have a parallax of one second of arc; this is about nineteen trillion miles. The distance of Alpha Centauri is a little over four light-years, or about one and a third parsecs. The average distance between the stars of the solar neighborhood is larger than this, being from five to ten light-years. An interesting fact brought out by a study of the nearest stars is the large number of very faint ones. Alpha Centauri is brighter than the sun, but the second and third nearest stars are both so faint that they are invisible to the naked eye. The fourth in distance is Sirius. In fact, of the twenty known stars nearest the sun, just half are visible, and probably there are other faint stars equally near which have not yet been discovered. Most of the naked-eye stars are far more brilliant than the sun, some of them several thousand times as bright, and we are liable to get the idea that our sun is among the fainter stars. If the stars in the solar neighborhood form a fair sample, however, as there is no reason to doubt, it is distinctly above the majority of stars in luminosity.

As we go away from the sun in a direction perpendicular to the plane of the Milky Way, the stars begin to thin out more and more, and at the distance of a thousand light-years we are near the limit of the stellar system in that direction. A more recent discovery is that the stars also thin out as we go toward the Milky Way, though not so rapidly. Consequently we see that the stars near the sun are not continuous with those of the Milky Way, as was formerly believed, but form a well-defined lens-shaped cluster, roughly 6000 light-years in diameter and 2000 light-years thick, with the sun near the center. Practically all the stars visible to the naked eye belong to this solar cluster, together with a much larger number of fainter stars.

Most of our present knowledge of the stars is limited to this inner system. The remarkable fact has been discovered that these stars are not moving entirely at random, as was formerly supposed, but show a preference for moving in two opposite directions in the plane of the Milky Way. Apparently the solar cluster is made up chiefly of two large groups of stars which are moving through each other with a relative velocity of 25 or 30 miles a second. A smaller third group of stars is practically at rest in the system.

In addition to these star-streams, as they are called, several smaller groups which move as units have been found. For example, about forty stars, mostly in the constellation Taurus, are moving parallel to each other with equal velocities through the solar cluster. Five of the stars of the Great Dipper and about ten others make up another group of the same sort. This cluster has the form of a flat disc, while the Taurus cluster is spherical. The Pleiades form a more compact group, in which the physical connection is more obvious.

Among recent discoveries one of the most important is the distinction between "giant stars" and "dwarf stars." According to present views of stellar evolution, a star begins its career as a vast mass of gas of low density and comparatively low temperature. As this gas contracts under the influence of its own gravitation, it can be proved mathematically that its temperature will rise, in spite of the fact that it is radiating heat all the time; heat is generated by the contraction faster than it is radiated away. This increase in temperature continues until the density becomes approximately that of a liquid; from that point on the heat is lost faster than it is generated, and the star gradually cools off. It follows that a star passes through any given temperature twice, once on the ascending and once on the descending branch of its temperature curve. For example, our sun has a surface temperature of about 6000 degrees Centigrade; it is more dense than water, a fact which indicates that it is on the descending branch. At some distant time in the past it had the same temperature on the upward branch; its diameter was then probably about ten times as great as at present, which means that its surface area was a hundred times as great as now; and since the temperature was the same, it was a hundred times as bright as now. In other words, it was then a "giant star," and is now a "dwarf star."

Stars are classified according to their spectra, which means practically according to their temperatures. The principal types are as follows. Type B (Orion type) consists of very hot bluish-white stars, in whose spectra the helium lines are prominent. Type A (Sirian type) consists of white stars with hydrogen lines the most conspicuous ones. Types F, G, and K may be grouped together and called the solar type; the stars are yellowish and their spectra contain numerous metallic lines; our sun belongs to type G. Type M is composed of red stars, whose spectra are dominated by the flutings due to titanium oxide.

If our present theories are correct, a star begins its course as a giant red star of type M, ascends the series toward type B as it becomes hotter, and descends as it cools off until it reaches type M again as a dwarf star. Probably only the most massive stars get as far as type B; the smaller ones reach their maximum and turn downward at type A or lower.

It was one of the giant red stars, Betelgeuse, which was selected when the first successful attempt was made three years ago to measure the diameter of a star. Since the temperature of a red star is comparatively low, a given area is much less luminous than the same area of a white or blue star; hence it was inferred that a very bright red star like Betelgeuse must be very large, and this proved to be the case. The diameter was found to be two or three hundred million miles.

Since the giant stars are so much more luminous than the dwarf stars, they are the ones which we see and study for the most part. Nearly all the stars visible to the naked eye are giant stars belonging to the solar cluster.

Beyond the edge of this solar cluster lies the Milky Way. As viewed from our position in space, it has the appearance of an irregular circular ring. Some astronomers have regarded it as actually being a ring of star-clouds, a sort of appendage to the solar cluster; this view would put our sun practically at the center of the known universe. The present tendency, however, is to regard the Milky Way as a flattened disc, extending far beyond the solar cluster, which need not be anywhere near the center.

Our views of the extent of the stellar universe have been vastly enlarged by the work of Dr. Harlow Shapley, now director of the Harvard Observatory, on globular clusters. These are great spherical clusters containing thousands of stars, condensed toward the center; about eighty or ninety of them are known. Some of the clusters contain Cepheid variables, and by means of these and other methods Shapley determined the approximate distances of all of them. The nearest one is about 20,000 light-years away, while the most distant one is at the enormous distance of 220,000 light-years. When their positions in space are computed, they are found to form a flattened ellipsoid whose equatorial plane coincides with that of the Milky Way; half of the clusters are on one side of this plane and half on the other. None of the clusters is nearer to the plane of the Milky Way than 4,000 light-years, a peculiar fact for which we can give no explanation. The center of this ellipsoid is in the direction of the constellation Sagittarius at a distance of about 60,000 light-years. In view of this symmetrical arrangement of the globular clusters with respect to the plane of the Milky Way, it seems probable that the Milky Way itself extends as far as the edge of the ellipsoid. If so, it is a flattened disc, about 300,000 light-years in diameter and 4,000 light-years thick, with its center 60,000 light-years from the sun. That the Milky Way actually does extend to distances as great as those suggested is indicated by the fact that among the faintest stars visible to us are Cepheid variables and other stars of high absolute brightness. From this standpoint we regard the Milky Way not so much as a single system, but as a system of systems, of which our solar cluster is only one. In addition to its clusters of stars, the Milky Way contains numerous nebulae, irregular masses of faintly luminous gas.

The motions of the globular clusters suggest a possible origin for some of the structures now included in the Milky

Way. Many of the globular clusters are moving toward the plane of the Milky Way at speeds which will bring them there in the course of a few million or a few billion years, periods which are short as compared with the lifetime of a star. It seems probable that they are oscillating back and forth across the Milky Way as a result of the attraction of the masses of stars there. The effect of such passages through the Milky Way would be gradually to break up the globular clusters into larger and more irregular systems, since the stars of the cluster would at least occasionally pass near enough to stars of the Milky Way to be deflected by the gravitational attraction of the latter. In this way the globular clusters would come to resemble the open clusters of which many are known in the Milky Way. It may be that the star-streams and the moving groups of stars in the solar cluster represent later stages in the dissolution of globular clusters. It is possible, then, to regard the Milky Way as largely made up of the disintegrated remains of originally independent systems.

Outside the Milky Way, apart from the globular clusters, which are evidently closely related to it, and the two Magellanic Clouds, which appear to be isolated clusters of stars and gaseous nebulae of the same general character as the Milky Way itself, the only known objects are the spiral nebulae, of which there are more than a million. These are rare in the parts of the sky near the Milky Way, but become more and more numerous the farther away we get from it. It looks as though they had a positive aversion to the Milky Way. This impression is confirmed by the spectroscope, which indicates that they are practically all moving away from the sun at high velocities, sometimes several hundred miles a second. The nature of these nebulae is unknown; when their light is analyzed with the spectroscope, it looks as if it came from a cluster of stars, and the general view is that they are made up of stars, although our telescopes are unable to resolve them. Their spiral form suggests that they are rotating about their centers, and some measurements indicate that this is the case. Attempts have been made to estimate their distances by various indirect means, but the results differ so widely that all we can say is that they are very far away.

The interesting theory has been propounded and has met with considerable favor that these spiral nebulae are "island universes," that is to say, systems of stars of the same general type as the Milky Way. According to this theory the Milky Way has a spiral form, which of course we cannot perceive because we are located near its center. This theory arose before Shapley's recent work on the globular cluster, when the dimensions of the Milky Way were thought to be much smaller than the figures I have named. If these figures are correct, it is highly improbable that the spiral nebulae are at all comparable in size to our system, and we have to give up the idea that there is any close similarity between them and the Milky Way. They probably represent an altogether different type of cosmic evolution.

In conclusion, I want to remark that no present-day discussion of the universe as a whole is complete which does not take into account Einstein's theory of relativity, which has important bearings on the nature of space and matter and gravitation. In Newtonian mechanics, space is infinite in extent, has the properties described by Euclidean geometry, and exists independently of any matter contained in it. According to the general theory of relativity, the geometrical properties of space are determined by the matter it contains; at a great distance from any matter space is approximately Euclidean, but it departs more and more from Euclidean properties as we approach a large body of matter. There are also difficulties connected with the view that space is infinite. Whether we accept Newton's theory or that of Einstein, we are compelled to believe that the total amount of matter in the universe is finite, since if the average density of matter were different from zero it would follow that the force of gravitation at every point would be infinite, a condition which we know is not true. Consequently the matter of the universe must be like an island lying in an empty region of infinite extent. This is an unsatisfactory conception, because the energy of the universe is constantly being radiated away to infinity and thus forever lost. To avoid these difficulties Einstein adopted the view that space is spherical in structure, that is, that it is in three dimensions what the surface of a sphere is in two dimensions, finite in extent, yet having no boundaries. In such a universe no energy would ever be lost; if it traveled away in a straight line it would ultimately return to its starting point, just as on the earth a man who traveled 25,000 miles along a great circle would find himself at his starting point again. Some other types of finite non-Euclidean spaces have also been suggested, but the subject is still in a highly speculative stage. Even if the universe is finite, the estimates which have been made of its size indicate that it is far larger than the dimensions of the Milky Way, so probably the results so far obtained with regard to the distribution of the stars will not have to be greatly modified, whatever theory of the universe as a whole is finally adopted.

P. M. BATCHELDER.

THE BROWN UNIVERSITY MATHEMATICAL PRIZE AT THE UNIVERSITY OF TEXAS

The annual competition in freshman mathematical topics in the form of a voluntary prize examination was held as usual at the University of Texas, early in the college year, namely on Saturday, the sixth of October, 1923. There were four questions and the time open was one hour. The first prize was won by Andrew Woods of Belton, prepared at the Belton High School, the second prize was won by Lovett Young, prepared at the Sour Lake High School, and the third prize was won by Roy Cotulla, prepared at the Cotulla High School.

The questions asked are as follows:

1. A crew rows at the rate of five miles per hour in still water. If it takes two hours to row one and four-fifths miles upstream and then back, what is the rate of the current?

2. A man and a boy working together do a piece of work in 8 days. If we assume that the man could do it alone in 12 days less time than the boy could, how long would it take each to do it?

3. Given two lines, l and m, and a point, P, on l. To construct a point Q on l whose distance from P shall be equal to its distance from m.

4. Given two points, P, Q, and a third point, R, which is not on the line passing through P and Q. Determine a point X by means of compasses alone (that is without using ruler) such that the line RX shall be parallel to line PQ.

ALBERT A. BENNETT.

University of Texas.

SOME REMARKS ON DIOPHANTUS OF ALEXANDRIA

From the time that the Roman armies finally put a formal end to the precarious nominal independence of Egypt in the year 30 B.C., until, nearly seven centuries later, the Arabs in their newly found Mohammedan fervor overran Rome's vast African possessions, Alexandria was the pride of imperial cultural prestige. Renowned for its wealth and commerce, it was no less famous as the abode of writers, artists and philosophers, as the site of the university where the young aristocracy of the Roman world absorbed the heritage of Greek learning and acquired the polish of Greek manners from direct contact with the most renowned scholars of the time. It must not be supposed that all the conspicuous names associated with that of Alexandria under Roman rule were those of teachers formally enrolled in the staff of the university. One must think of the city not so much as housing a special organized body of instructors as being a general meeting place of educated men, a center attracting the ambitious student, encouraging learning and research, and in turn radiating over the civilized world such concepts and theories as found lodgment in the practical Roman mind.

This long period for all its advantages showed little mathematical progress. Scholars were largely content with systematizing and emendating the great works of original genius handed down from a more glorious past, and even the mere routine of instruction became hampered with ever-recurrent restrictions and eventually practically terminated by the dogmatic fanaticism of a religious sectarian ardor which persecution, popular mysticism and violent political upheavals could hardly fail to have aroused in the minds of many unenlightened demagogues.

Although biographical histories of mathematics mention not a few names of this period that have come down to us through the ages, many of the persons thus remembered can mean little to most of us. Some tracts of doubtful value, the unproved statements of a few propositions, or perhaps merely a chance reference to the existence of such a name among the celebrities of the day, such are in most cases about all the extant evidence for the claims even of the best of these departed scientists. Conspicuous exception must be made of Ptolemy whose astronomical work continued the practically unquestioned authority for a millenium, and whose mathematical achievements were noteworthy. Other names that occur to one are those of Menelaus, celebrated for a familiar theorem of the ratio of segments upon the sides of a triangle, and Pappus, several of whose theorems seem like the introductory steps of a new and far-reaching geometrical theory—a theory which remained thus merely suggested until the opening of the nineteenth century. In the midst of all this waste of pedagogical reiteration lighted only at rare intervals by brief and disappointing flashes of Greek intellect, there appeared a mathematician of a different type (there is some reason for supposing that he was not even a Greek)—trifling perhaps in many of his problems, and, it has been urged, with few general rules and many special unproved tricks of narrow application, and yet beyond doubt, markedly original in problem, method, and detail—such was Diophantus of Alexandria.

In a time when geometry of the narrowest Euclidean type was almost the sole subject of mathematical study, Diophantus investigated arithmetic, number theory and algebra. He made consistent use for the first time in mathematical history of abbreviations thus rendering visually more comprehensible the wordy rhetorical language in which mathematical enunciations had always been expressed. This was a first undirected step toward the purely symbolic notation that is a familiar feature of modern algebraic language. Diophantus studied each problem on its own merits. Even a general method of procedure was examined in the light of a particular numerical illustration. Any peculiarity in the situation that could be used to simplify the steps was seized with an energetic ingenuity that still causes the critics to marvel. One is constantly astonished at his adroit handling of the problems he proposes but the casual reader might be excused at first for questioning whether such neat solutions would have been forthcoming if some one else had had the privilege of proposing the problems for solution. However, one soon comes to respect both the temerity and the persistent success of this ancient writer.

I will give in modern notation and with explanatory remarks only suggested by the text, five problems that are fairly typical of the methods of Diophantus. These are the ones which were selected by Ball in his well known History of Mathematics.

(1) Find four numbers the several sums of these taken three at a time being given, say 20, 22, 24, 27. (One may remark that this problem in linear equations is given in special form not because the more general problem would present to us serious difficulties, but because a special artifice can be used in this case by which a single letter, x, may be made to provide the entire solution.) Diophantus takes x to denote not any one of the desired unknowns but rather their sum, the desired answers being therefore given by the expressions, $x-20$, $x-22$, $x-24$, $x-27$. Hence $x=(x-20)+(x-22)+(x-24)+(x-27)$, or $x=4x-93$, $x=31$. The numbers desired are therefore 11, 9, 7, and 4.

(2) To express a number which is known to be the sum of two squares as the sum of two other squares, say, 13, which is 2^2+3^2. Diophantus uses more than the fact that the given number is of a special form, namely expressible as the sum of two squares; he uses the actual value of the two separate known squares, to eliminate the sum. It would appear natural to us probably to take the unknowns to be x and y so as to lead to the equation $(x^2-2^2)+(y^2-3^2)=0$. The solution is to be in rational numbers and there is clearly an extra algebraic parameter at our disposal. Diophantus always contrives to get along with but one explicit unknown at a time, although a parameter is sometimes carried along in the discussion. In this case Diophantus takes one of the unknowns which is to be squared, in the form, $2+x$. The other is then less than 3, and is taken in the form, $3-mx$, where m is a parameter. We then have $(2+x)^2+(3-mx)^2=2^2+3^2$, whence $4x-6mx+x^2+m^2x^2=0$. The solution, furnished by $x=0$, does not

satisfy the condition of providing a new representation, and is therefore rejected, leaving, $x(m^2+1)=6m-4$. For each value of *m*, a solution results. For $m=1$, the original representation is reobtained with only the order of terms interchanged. Diophantus chooses therefore $m=2$, giving finally, 1/25, and 324/25 as the required squares.

(3) Find two squares so that each plus their product is a square. Using x^2 for one of the two given squares, we have (x^2+1) times the other square is a square, so that x^2+1 must be itself a square. For more convenient manipulation, Diophantus introduces a square factor in numerator and denominator and thinks of this as $(4x^2+4)/4$. This he identifies arbitrarily with $(x^2-4x+4)/4$ leading to the determination of x as 3/4. The second quantity can now be called the unknown, and we would be consistent with the spirit of these investigations were we to call this second unknown from now on, x. However, it will be easier for us to call it y. For the same reason as before y^2+1 must be a perfect square. The expression comes to us in the form, $(9y^2/16)+(9/16)$, so that it is not necessary this time to introduce an arbitrary factor. The numerator, $9y^2+9$ must be a perfect square. Instead of comparing this with a quadratic expression having the same constant term as on the last occasion, Diophantus varies the procedure by identifying it with a quadratic square having the same square term. He selects, in fact, for the fraction the expression, $(9y^2/16)-(3y/2)+1$, whence $y=7/24$. The squares required are 9/16 and 49/576. Diophantus does not attempt to verify any of his solutions by testing the effect of substituting the values obtained, and in this respect is followed by most modern texts. He does not examine the character of the problem with a view to determining the total number of possible solutions. The distinction among problems as to degree and as to their being determinate or indeterminate had to await the development of algebraic notation. It was many centuries before any one dreamed of the possibility of any such classification. The Greek notation by which letters were used to denote the familiar natural numbers was particularly unfavorable to a systematic classifi-

cation in the various disciplines connected with operations with numbers. Even the idea of utilizing two distinct unknowns co-ordinately in a pair of simultaneous equations, may never have occurred to Diophantus, but if he had considered such an experiment he might well have concluded that such a measure would have proved inextricably confusing to any of his readers if not to himself.

(4) To find a right triangle with rational numbers for its sides the length of the bisector of one of whose acute angles is also rational. Let the triangle have vertices, A, B, C, of which C is the right angle. Let AD be the bisector of the angle BAC. Then $BD/DC=BA/AC$. Also $AD^2=AC^2+DC^2$. Diophantus selects the ratios of the sides of the small triangle, ACD, arbitrarily to be in the form, 5:4:3. In particular, he takes $DA=5x$, $AC=4x$, $CD=3x$, where x is still to be determined. Since this makes $AC:CD=4:3$, we shall have also $AB:BD=4:3$. We have therefore that AB is the same multiple of 4 that BD is of 3. The problem is homogeneous in the sense that if one solution is known any number of others can be found by merely multiplying all linear dimensions by a common rational factor. There is no objection therefore in taking one of the sides of the triangle ABC as arbitrary. Diophantus therefore chooses for the sake of simplicity, BC to be 3. This makes $BD=3-3x$ and $AB=4-4x$. For the triangle ABC we now have $(4-4x)^2=3^2+(4x)^2$. From this expression the quadratic terms drop out, so that x is rational being in fact$=7/32$. To obtain an integral solution, each linear dimension is multiplied by 32, giving, 28, 96, 100, for the sides, and 35 for the bisector which meets the side 96.

(5) A man buys x measures of wine, some at 8 drachmae a measure and the rest at 5. He pays for them a square number of drachmae such that if 60 be added to it, the resulting number is x^2. Find the number he bought at each price.

The price paid is x^2-60. Had he paid 8 drachmae a measure for the whole lot the cost would have been 8x, but this is in excess of the actual expenditure. In a similar manner 5x is less than the actual cost. Thus $8x>x^2-60>5x$. For very large values of x the first inequality must break down while for x positive but less than the square root of 60 the second inequality must fail. Trying successive integers to determine if possible a range of validity, we find that $11<x<12$. Further we have the condition that $x^2—60$ is a perfect square. Its square root must be less than x and may therefore be denoted by $x—m$. We have then, $x^2—2mx+m^2=x^2—60$, or $x=(m^2+60)/(2m)$. We have therefore using this value of x, $11<(m+60)/(2m)<12$, or $22m<m^2+60<24m$. Again the first inequality fails for very large values of m while the second cannot hold for value of m sufficiently near to zero. Testing successive integers as before we find that $19<m<21$. Any admissible value of m will give a solution, so that Diophantus chooses the integral solution of these inequalities, $m=20$. This determines x as $11\frac{1}{2}$ and fixes the total cost at $72\frac{1}{4}$ drachmae. The problem, however, is not yet completed. Let y denote the number of drachmae paid for the whole amount of the cheaper wine bought. The number of measures of this wine purchased will then be $y/5$. The cost of the dearer wine is then $72\frac{1}{4}—y$ and the number of measures bought will be expressed by an eighth of this difference. But the total amount bought was $11\frac{1}{2}$ measures. Thus $(y/5)+[(72\frac{1}{4}—y)/8]=11\frac{1}{2}$, whence $9^1/_{32}+{}^{3y}/_{40}=11\frac{1}{2}$ or $3y/40=2^{15}/_{32}$ or $y=5$ times $79/12$. Thus the number of measures of wine at 5 drachmae is $79/12$ or $6^7/_{12}$, and the remainder or $11\frac{1}{2}—6^7/_{12}=4^{11}/_{13}$ is the number of measures bought at 8 drachmae.

During later centuries the work of Diophantus was practically unknown only to be rediscovered after algebraic methods made many of his simpler problems of little further interest. His fame is deservedly perpetuated, however, in the name given to the subject of indeterminate numerical equations to be solved in integers or rational numbers, namely Diophantine Analysis.

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WHY ANOTHER METHOD OF COMPLETING THE SQUARE?

I once thought that it was useless to criticize methods in our adopted text which I omitted in my teaching, for of course I thought that no one used methods which are duplicates or unnecessary burdens to students. But I have decided differently. I consider the material given on pages 268 and 270 in Wentworth's New School Algebra an additional and useless tax. "Another method of completing the square." Why another method? Is not one sufficient? "The square of a half the coefficient of x added to both sides of the equation provided the coefficient of x is 1,"[1] suffices for completing the square of any complete quadratic equation whether numerical or literal.

If some teacher of mathematics away off in some school corner of Texas thinks the writer is "crazy" as he never used any method except the one recommended, I hope that he does. But in our mathematical corps, as highly as we rate ourselves, there have been those among my colleagues who have tried to teach three or four different devices for completing the square. Thus they hardly allow the pupil time to grasp one method, the first one given on page 266, and its accompanying problems, before turning over to page 268 to another method with a set of the same type of problems, and next to page 270 to still a special case for completing the square of a complete quadratic equation—when all the problems regardless of page or special type, could have been solved by the first method given.

I find in our text these special rules for completing the square of a complete quadratic equation under certain specific conditions:

"If the coefficient of x^2 is 4, 9, 16 or any other perfect square, we may complete the square by adding to each side the square of the quotient obtained from dividing the second term by twice the square root of the first term,"[2] and,

[1] *Wentworth's New School Algebra*, pp. 266, 293.
[2] *Wentworth's New School Algebra*, p. 268.

"If the coefficient of x^2 is not a perfect square, we may multiply the equation by a number that will make the coefficient of x^2 a perfect square."[3]

"If a complete quadratic is multiplied by four times the coefficient of x^2, fractions will be avoided."[4]

I am not trying to avoid fractions, but still that is another subject. And again we find this, "If the coefficient of x is an even number, we may multiply by the coefficient of x^2, and add to each member the square of half the coefficient of x in the given equation."[5]

I wonder if it is good mathematics to bother over whether or not the coefficient of x^2 is a perfect square or the coefficient of x is an even number, or the equation is multiplied by four. Perhaps the pupil who recognizes the method when the coefficient of x^2 is 9, would be lost entirely with some other coefficient.

Since I never use them, it is true that I know little about these last methods quoted from pages 268 and 270. But I am looking them over now so as to tell how needless I think they are. I employ one method which applies invariably granting that the coefficient of x^2 is any rational number. for example:

Compute the positive root of the following equations to the nearest hundredth.[6]

$$4x^2-9x-11=0$$

$$x^2-\frac{9}{4}x=\frac{11}{4}$$

$$x^2-\frac{9}{4}x+\left(\frac{9}{8}\right)^2=\frac{11}{4}+\left(\frac{9}{8}\right)^2$$

$$=\frac{257}{64}$$

$$x-\frac{9}{8}=\pm\frac{\sqrt{257}}{8}$$

$$x=3.12$$

$$\text{and } x=-.878 \text{ (negative root)}$$

[3]*Ibid.*

[4]*Wentworth's New School Algebra*, p. 270.

[5]*Ibid.*

[6]From College Board Examination, 1921.

Or take one of the equations in the text which is worked by a special case and work it by the one method recommended, then compare steps in the two solutions.[7]

$$3x^2+x=20$$
$$x^2+\frac{4}{3}x=\frac{20}{3}$$
$$x^2+\frac{4}{3}x+\frac{4}{9}=\frac{20}{3}+\frac{4}{9}$$
$$=\frac{64}{9}$$
$$x+\frac{2}{3}=\pm\frac{8}{3}$$
$$x=2,\ \text{or}\ -3\tfrac{1}{3}$$

Furthermore, omitting the explanations on pages 268-270 necessitates another change in text; that is, paragraph 299, page 278, in the method of solving the general quadratic equation to derive the formula. The general quadratic equation may be assigned to the class for solution by this one method for completing the square just as one would assign any literal quadratic equation consisting of three terms.

$$ax^2+bx+c=0$$
$$x^2+\frac{b}{a}x=-\frac{c}{a}$$
$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$
$$=\frac{b^2-4ac}{4a^2}$$
$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2ac}$$

I am not urging the superiority of completing the square

[7]Par. 296, p. 270.

over factoring or the use of the formula in solving a complete quadratic equation, but since we must teach a method by which one can complete the square, let the method be one that is applicable in every case. We must know how to complete the square, since it is convenient for imaginary or surd roots, and for deriving the formula, but why waste our limited time teaching four devices when one will serve every purpose?

I have worded to suit my purpose what is found in step one, paragraph 296, page 266, quoted here. "To solve a complete quadratic equation, the first step is to add to both members the square of half the coefficient of x. This is called completing the square. The second step is to extract the square root of each member of the resulting equation. The third step is to reduce the two resulting simple equations."

This, however, is all that I require learned:

1. If the coefficient of x^2 is other than one, divide both sides of the equation by the coefficient of x^2.
2. Then, add to both sides of the equation the square of half the coefficient of x.

I agree that in solving a complete quadratic my preference is to factor; second, to use the formula; and last to complete the square, but when the last is used, insist on the above method which will apply generally.

ALMA HOUSTON.

MATHEMATICS AS A LANGUAGE

The teacher who has the privilege of initiating a pupil into the mysteries of algebra or geometry has a pleasurable task as well as an opportunity to inspire a wholesome interest in the whole field of mathematics.

There is real joy in observing students grasp, for the first time, the idea that a mathematical discussion is a composition in mathematical language and should not consist of disjointed clauses, but of clear mathematical sentences, so that any one familiar with the language may get the idea to be conveyed as readily as if he were reading an English composition. The ease with which a pupil acquires a mathematical vocabulary depends upon his ability to read English intelligently. The language is acquired just like any other language and there is need for translation back and forth between the symbolic language of mathematics and the more prolix mother tongue. The student must be taught from the beginning that mathematics, especially algebra, is very dependent upon its symbolism. However, this should not make it a difficult subject. On the contrary, by use of symbols, a complete thought can be put into a form so concise as to be comprehended at a glance. When an equation does not express the thought of the problem, the trouble is not with the language of mathematics. It simply means that the pupil does not know the language of the subject. He needs a vocabulary he can use intelligently. Pupils come to us from grammar school who do not understand the significance of the equality sign. They have used it so commonly from the beginning of their school career that it has become a meaningless convenience to them. Likewise, the words equation, identity, root, reciprocal, ratio, power, satisfy, etc., all belonging to the vocabulary of arithmetic, are not more difficult than other words used by students of average ability, and yet they are seldom used readily and intelligently in the classroom.

The study of mathematics gives the student an opportunity of observing the necessity of stating facts with care. He learns that there is no compromise with an equation.

It is either solved or it is not. If we are insistent upon accurate expression of thought in the language of mathematics, our pupils will enjoy more than anything else the lessons on the interpretation of thought, and they will even become quite skillful in expressing symbolically the finer shades of meaning.

The pupil should be made to see that the training which he receives in working with the symbols of mathematics will be of use to him after he leaves school. Much of the world's work requires the constant mastery of symbols. Only those persons in the least remunerative and least desired positions work entirely with actual things. Every business uses the graph to a great extent in giving their statistical reports. The girl at the telephone board, the man in the signal tower, the president of a railroad or other great corporation—each is directing, by means of symbols, the activities of gigantic industries.

The department of mathematics is a field for excellent service. There is no doubt but that a teacher's success is readily measured in exact terms by the skill of his pupils in the mastery of such definite tasks as are presented in the algebra and geometry courses. So, if we, as teachers of elementary mathematics, would do away with dead formalism, and teach pupils to get a clear-cut thought, to express it in the language of mathematics, and then to write it correctly in symbolic form, our pupils will leave school feeling that they had gained more than a mere facility in juggling symbols. When we make our pupils feel the need of a deeper purpose and a more sane and practical view of mathematics in their ranks, then and only then shall we cease hearing of so many failures in the higher mathematics.

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Reference: Young's "The Teaching of Mathematics."

A NEW METHOD IN TEACHING GEOMETRY

Much interest is being manifested both at home and abroad in the teaching of geometry as it has been done in the high school of Fresno, California, for the past two years. During the summer session of the University of California, there was much discussion of this and kindred topics by a group of mathematics teachers, and profitably so, for the method has many good points and is being adopted by several other high schools of California. To quote Mr. Austin of the mathematics department of the Fresno High School, "the scheme is the result of an effort to make the study of geometry both valuable and interesting to students and to make the subject matter taught and the method of attack conform to modern ideas of education and life" and no one in the department "thinks of dropping back into the old time-honored plan."

The usual classroom of rows of desks is replaced by a room equipped with drawing table and chairs and each student has his drawing board, T-square, a triangle, a scale, a protractor, a compass, dividers, thumb tacks, a folder containing sheets of drawing paper, and a copy of the *Manual* instead of the usual textbook. The student must, first, make constructions according to specific directions; second, take measurements and perform computations; third, state the conditions of construction and the apparent conclusion; fourth, give the usual formal proof to establish the truth in general; fifth, state the general truth in the form of a proposition. In this manner he is to develop concepts of the fundamental principle of geometry. The claim that when the student makes his own figure, following specific direction, step by step, he comprehends the whole as a logical process, is well grounded and supplies a need felt by mathematics teachers who deserve their name. An Illinois teacher claims practically the same results by teaching her class without any textbook at all. The notebook of the student as it was made served him as a textbook.

This article only hints at the details of the plan, but full description is given by Mr. Austin himself in "A Dream

Come True" in the October number of Volume XXI, 1921, "School Science and Mathematics," and is well worth consideration. Another article by Mr. Austin on the same subject is "Our Geometry in Egypt and China," from the *Mathematics Teacher*, Volume XVI, February, 1923, No. 2.

The last article mentioned above tells of the use of the new method by a professor of mathematics in Assint College, Assint, Egypt. Many other foreigners have sent inquiries and comments.

MYRTLE C. BROWN.

THE STRAIGHT EDGE

A "Sorter" Catechism

Can you answer the question: "Why should I study Geometry?"

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Do you feel that as a teacher you ought to know it?

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Do you regard this perennial question of the student as a proper one?

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Are you pleased or annoyed when it is presented to you?

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Which would you be if you felt sure that you had a good and sufficient answer—one that satisfied you, for instance?

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How much time have you spent seriously trying to find a good answer?

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Do you regard the statement that "A study of geometry improves the mind" as a sufficient answer?

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Resolved: I am going to find the answer to this question and will not stop till I have one that satisfies me and that I believe ought to satisfy my students.

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